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# Edge waves on a gently sloping beach

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Edge waves of frequency $\omega$ and longshore wavenumber $k$ in water of depth $h(y) = h_1 H(\sigma y/h_1), 0 \leqslant y < \infty$, are calculated through an asymptotic expansion in $\sigma/kh_1$ on the assumptions that $\sigma \ll 1$ and $kh_1 = O(1)$. Approximations to the free-surface displacement in an inner domain that includes the singular point at $h = 0$ and the turning point near $gh \approx \omega^2/k^2$ and to the eigenvalue $\lambda \equiv \omega^2/\sigma g h$ are obtained for the complete set of modes on the assumption that $h(y)$ is analytic. A uniformly valid approximation for the free-surface displacement and a variational approximation to $\lambda$ are obtained for the dominant mode. The results are compared with the shallow-water approximations of Ball (1967) for a slope that decays exponentially from $\sigma$ to 0 as $h$ increases from 0 to $h_1$ and of Minzoni (1976) for a uniform slope that joins $h = 0$ to a flat bottom at $h = h_1$ and with the geometrical-optics approximation of Shen, Meyer & Keller (1968).

## 1. Introduction

The classical edge wave is that of Stokes (1846) on a uniformly sloping beach, $z = -\sigma y$, for which the free-surface displacement and dispersion relation are given by

$$\zeta(x,y,t) = a\,\mathrm{e}^{-\nu k y}\cos(kx-\omega t) \quad (k|a| \ll 1), \tag{1.1}$$

and

$$\omega^2 = \lambda \sigma g k, \tag{1.2}$$

with

$$\lambda = \nu = (1+\sigma^2)^{-\frac{1}{2}} = \cos\beta \quad (\beta \equiv \tan^{-1}\sigma). \tag{1.3}$$

This Stokes edge wave is the dominant member of a discrete set for which $\lambda\sigma = \sin(2n+1)\beta$ and $n = 0, 1, \ldots$ up to the largest integer for which $(2n+1)\beta < \frac{1}{2}\pi$ (Ursell 1952).

I consider here the generalization of these results for uniform slope to a gently sloping beach, $z = -h(y)$, on the assumptions that $h(y)$ is smooth (at least to the extent that the derivatives that occur explicitly in the subsequent development exist) and has the limits

$$h \downarrow \sigma y \quad (y \downarrow 0), \quad h \sim h_1 \quad (y \uparrow \infty), \tag{1.4$a, b$}$$

where

$$0 < \sigma \ll kh_1 \equiv \kappa = O(1). \tag{1.5$a, b$}$$

It is expedient to specify $h(y)$ in the form

$$h = h_1 H(\eta), \quad \eta = \sigma y/h_1, \tag{1.6$a, b$}$$

(so that $H \downarrow \eta$ as $\eta \downarrow 0$ and $H \sim 1$ as $\eta \uparrow \infty$) and to introduce

$$\epsilon \equiv \frac{\sigma}{kh_1} = \frac{\sigma}{\kappa} \ll 1 \tag{1.7}$$

as the ratio of the wave scale $1/k$ to the beach scale $h_1/\sigma$.

I begin my analysis in §2 by invoking an earlier (Miles 1985) formulation for waves in water of variable depth and posing the free-surface displacement in the form (cf. (1.1))

$$\zeta = aZ(y)\cos(kx-\omega t), \tag{1.8}$$

(or, with trivial changes, $\zeta = aZ(y)\cos kx \cos \omega t$) to obtain the differential equation for $Z(y)$. This equation has singularities at $y = 0\,(h = 0)$ and $y = \infty$ and a turning point that, for $\lambda = O(1)$, is near $gh = \omega^2/k^2$, in consequence of which the limit $\epsilon \downarrow 0$ leads to a singular perturbation problem for which the inner and outer lengthscales are $1/k$ and $h_1/\sigma$.

In §3, I develop an inner expansion for which the independent variable is $kh/\sigma$. The first approximation to $\omega$ is provided by the solution for a uniform slope, while the second approximation is given by

$$\frac{\omega^2}{gk} = (2n+1)\left(\frac{\mathrm{d}h}{\mathrm{d}y}\right)_0 + (n^2+n+\tfrac{1}{2})\,k^{-1}\left(\frac{\mathrm{d}^2h}{\mathrm{d}y^2}\right)_0 + O(\sigma^3)\,[\sigma \downarrow 0, \kappa = O(1)]. \tag{1.9}$$

The corresponding expansion of $Z(y)$ is not uniformly valid as $y \uparrow \infty$, but it suffices for many applications, particularly for the higher modes.

The turning point is relatively unimportant for the dominant mode, which is non-oscillatory and has no zeros in $0 \leqslant y < \infty$. It then proves possible to obtain a single, uniformly valid expansion of $Z(y)$ (§4) and a rather efficient variational approximation for the dominant eigenvalue (§5). These results should be useful in various oceanographic contexts, in which only the dominant mode is likely to be significant.

Finally, in §6, I compare the results of §§4 and 5 with the shallow-water ($\kappa \ll 1$) approximations of Ball (1967) for an exponentially decaying slope and of Minzoni (1976) for a discontinuous (from $\sigma$ to 0 at $y = h_1/\sigma$) slope.

The problem of edge waves on a gently sloping beach also has been attacked by Shen, Meyer & Keller (1968) using Keller's (1958) geometrical-optics approximation for gravity waves in water of gradually developing depth. This approximation (like WKB approximations in general) is not expected to be accurate for the dominant mode; moreover, Shen *et al.* ignore the singularity at $h = 0$†, which might be thought to affect their result for all modes. I consider their approximation in the Appendix and find that it reproduces the first term in (1.9) exactly and yields $n^2+n+\frac{1}{4}$ in place of $n^2+n+\frac{1}{2}$ in the second term. It fails at $O(\epsilon^2)$, at least for Ball's (1967) profile, but this appears to reflect the intrinsic order of Keller's approximation rather than the neglect of the singularity at $h = 0$.

## 2. Formulation

The linearized problem for monochromatic surface waves on water of variable depth $h$ admits a solution for the velocity potential in the form (Miles 1985)

$$\phi = Re\{\mathrm{e}^{-\mathrm{i}\omega t}(\cosh \ell z + K\ell^{-1}\sinh \ell z)\,\Phi(x,y)\}, \tag{2.1}$$

where

$$\ell^2 = -\partial_x^2 - \partial_y^2, \quad K = \omega^2/g, \tag{2.2}$$

$$\nabla\cdot\left[\left\{\frac{\sinh \ell h}{\ell} + K\left(\frac{1-\cosh \ell h}{\ell^2}\right)\right\}\nabla\Phi\right] + K\Phi = 0, \tag{2.3}$$

† Keller (1958) explicitly recognizes, and subsequently (1961) accommodates, this singularity in his treatment of the continuous spectrum. See also Shen & Keller (1975).

and $\mathscr{k}$ operates only on $\Phi$ (and not on $h$). Posing the free-surface displacement in the form (1.8) and invoking the free-surface boundary condition $\phi_t = -g\zeta$, we have

$$\Phi(x,y) = (ga/\mathrm{i}\omega)\,\mathrm{e}^{\mathrm{i}kx}Z(y), \quad \mathscr{k}^2 = k^2 - \partial_y^2. \tag{2.4$a,b$}$$

Substituting (2.4$a$) into (2.3) and re-arranging, we obtain

$$(K\cosh \mathscr{k}h - \mathscr{k}\sinh \mathscr{k}h)\,Z + h'(\cosh \mathscr{k}h - K\mathscr{k}^{-1}\sinh \mathscr{k}h)\,Z' = 0, \tag{2.5}$$

wherein $\mathscr{k}$ operates only on $Z$, and $' \equiv \mathrm{d}/\mathrm{d}y$.

The edge-wave boundary conditions are

$$Z = 1 \quad (y=0), \quad Z \to 0 \quad (y \uparrow \infty), \tag{2.6$a,b$}$$

which can be satisfied for (and only for) a discrete set of

$$\lambda \equiv \omega^2/\sigma g k. \tag{2.7}$$

There also is a continuous spectrum, for which (2.6$b$) is replaced by the specification of an incoming wave and for which an analysis similar to that of §4 below yields an approximation equivalent to that of Keller (1958, 1961).

## 3. Inner expansion

An appropriately scaled inner variable for the domain between the singularity at $h=0$ and the neighbourhood of the turning point at $kh = \omega^2/gk$ is

$$\xi = \sigma^{-1}kh(y) = \epsilon^{-1}H(\eta). \tag{3.1}$$

Transforming (2.4$b$) and (2.5) and expanding in powers of $\sigma$, we obtain

$$\{\xi\mathscr{D}^2 + H'\mathscr{D} + \lambda - \xi - \tfrac{1}{2}\sigma^2[\tfrac{1}{3}\xi^3(\mathscr{D}^2-1)^2 + H'\xi^2(\mathscr{D}^2-1)\,\mathscr{D} + \lambda(\xi^2\mathscr{D}^2 + 2H'\xi\mathscr{D} - \xi^2)] + O(\sigma^4)\}\,Z = 0, \tag{3.2}$$

where

$$\mathscr{D} \equiv H'\,\partial\xi, \quad H' = 1 + \epsilon H_0''\,\xi + \tfrac{1}{2}\epsilon^2(H_0''' - H_0''^2)\,\xi^2 + \ldots, \tag{3.3$a,b$}$$

and $H_0^{(n)} \equiv \mathrm{d}^nH/\mathrm{d}\eta^n$ at $\eta = 0$. Substituting (3.3) and the expansions

$$\lambda = \lambda_0 + \epsilon\lambda_1 + \ldots, \quad Z = Z_0(\xi) + \epsilon Z_1(\xi) + \ldots, \tag{3.4$a,b$}$$

into (3.2) and equating powers of $\epsilon$, we obtain

$$(\xi Z_0')' + (\lambda_0 - \xi)\,Z_0 \equiv \mathscr{L}Z_0 = 0, \tag{3.5$a$}$$

$$\mathscr{L}Z_1 = -\lambda_1 Z_0 + H_0''[2\xi(\lambda_0-\xi)\,Z_0 - \xi Z_0'], \ldots \tag{3.5$b$}$$

Identifying (3.5$a$) as Laguerre's equation, we obtain

$$\lambda_0 = 2n+1, \quad Z_0 = \mathrm{e}^{-\xi}L_n(2\xi) = \frac{\mathrm{e}^{\xi}}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^n \xi^n\,\mathrm{e}^{-2\xi} \quad (n = 0,1,2,\ldots). \tag{3.6$a,b$}$$

$\lambda_1$ then may be determined from the requirement that the right-hand side of (3.5$b$) be orthogonal to $Z_0$. Multiplying (3.5$b$) throughout by $Z_0$, invoking the identity

$$Z_0\mathscr{L}Z_1 = [\xi Z_0^2(Z_1/Z_0)']', \tag{3.7}$$

integrating over $(0,\infty)$, and invoking the null condition at $\xi=\infty$, we obtain

$$\frac{\lambda_1}{H_0''}=\frac{\int_0^\infty [2\xi(\lambda_0-\xi)Z_0^2-\xi Z_0 Z_0']\,\mathrm{d}\xi}{\int_0^\infty Z_0^2\,\mathrm{d}\xi}. \tag{3.8}$$

Integrating $\xi Z_0^2$ and $\xi^2 Z_0^2$ with the aid of Buchholz's (1953) §12(21) and integrating $\xi Z_0 Z_0'$ by parts, we reduce (3.8) to

$$\lambda_1=(n^2+n+\tfrac{1}{2})H_0''. \tag{3.9}$$

Combining this last result with $\lambda_0=2n+1$ and invoking (1.6) and (2.7), we obtain (1.9).

## 4. Uniformly valid expansion for dominant mode

The approximation (3.4) is $O(\mathrm{e}^{-1/\epsilon})$ as $y\uparrow\infty$ but does not exhibit the correct exponential decay. This difficulty may be overcome by matching the inner expansion to an appropriate outer expansion; however, this is of limited interest for the higher modes and unnecessary for the dominant mode, for which a uniformly valid approximation may be obtained in the form

$$Z(y)=\exp\{-\nu ky+\int_0^\eta F(\eta)\,\mathrm{d}\eta\}, \tag{4.1}$$

where $\eta$ is defined by (1.6), and $\nu$ is determined by

$$\nu=(1-\mu^2)^{\frac{1}{2}},\quad \mu\tanh(\kappa\mu)=K/k=\lambda\sigma, \tag{4.2a, b}$$

which implies the satisfaction of (2.4) for $h=h_1$. Positing the expansions

$$\lambda=\lambda_0+\epsilon\lambda_1+\dots,\quad F=F_0+\epsilon F_1+\dots, \tag{4.3a, b}$$

expanding (4.2$b$) about $\mu=0$ to obtain

$$\mu^2=1-\nu^2=\epsilon\lambda+\tfrac{1}{3}\sigma^2\lambda^2+\dots, \tag{4.4}$$

substituting into the corresponding expansion of (2.5), and equating powers of $\epsilon$, we obtain

$$2HF_0=\lambda_0(1-H)-H', \tag{4.5a}$$

and

$$2HF_1=\lambda_1(1-H)+\tfrac{1}{2}H'+(H+H')F_0+H(F_0'+F_0^2)$$
$$+\kappa^2[H(1-\tfrac{1}{2}H)(H'-\tfrac{1}{3}+\tfrac{1}{3}H)+H^2(1-H'-\tfrac{2}{3}H)F_0-\tfrac{2}{3}H^3(F_0'+F_0^2)],\dots \tag{4.5b}$$

Requiring $F_0$ and $F_1$ to be bounded at $\eta=0$, which determines $\lambda_0$ and $\lambda_1$, respectively, we obtain

$$\lambda_0=1,\quad F_0=\frac{1-H'-H}{2H}, \tag{4.6a, b}$$

$$\lambda_1=\tfrac{1}{2}H_0'',\quad F_1=\frac{(1-H')^2-H^2}{8H^2}+\frac{H_0''(1-H)-H''}{4H}+\tfrac{1}{6}\kappa^2(2H'+HH''). \tag{4.7a, b}$$

We remark that $F_0$ and $F_1$ are finite at $\eta=0$ and vanish as $\eta\uparrow\infty\,(H\sim 1)$, by virtue of which the expansion (4.3$b$) is uniformly valid.

## 5. Variational approximation (dominant mode)

The variational form (Miles 1985)

$$K = \frac{\langle \nabla \Phi^* \cdot (\ell^{-1} \sinh \ell h \, \nabla \Phi) \rangle}{\langle \Phi^* \Phi \rangle + \langle \nabla \Phi^* \cdot [\ell^{-2}(\cosh \ell h - 1) \, \nabla \Phi] \rangle}, \tag{5.1}$$

where $\langle \; \rangle$ signifies an average over the free surface, is stationary with respect to joint variations of $\Phi$ and $\Phi^*$ about the respective solutions of (2.3) and its adjoint. Substituting the trial functions

$$\Phi = \Phi_0 \, e^{ikx - \nu k y}, \quad \Phi^* = \Phi_0^* \, e^{-ikx - \nu k y}, \tag{5.2a, b}$$

where $\nu$ is determined by (4.2), into (5.1), invoking (1.6), integrating by parts, and dividing the result by $\sigma k$, we obtain

$$\lambda = \frac{\epsilon^{-1}(1+\nu^2) \displaystyle\int_0^\infty e^{-2(\nu/\epsilon)\eta} \cosh [\kappa \mu H(\eta)] \, H'(\eta) \, d\eta}{1 + \kappa \mu^{-1}(1+\nu^2) \displaystyle\int_0^\infty e^{-2(\nu/\epsilon)\eta} \sinh [\kappa \mu H(\eta)] \, H'(\eta) \, d\eta}. \tag{5.3}$$

The integrals in (5.3) may be expanded in powers of $\epsilon$ through repeated integration by parts (as in the asymptotic expansion of Laplace transforms). Carrying this expansion through $O(\epsilon^2)$ and invoking (4.4), we obtain

$$\lambda = 1 + \tfrac{1}{2}\epsilon H_0'' + \tfrac{1}{4}\epsilon^2(\tfrac{1}{2} + H_0'' + H_0''' - 2\kappa^2) + O(\epsilon^3). \tag{5.4a}$$

and

$$\nu = 1 - \tfrac{1}{2}\epsilon - \tfrac{1}{4}\epsilon^2(\tfrac{1}{2} + H_0'' + \kappa^2) + O(\epsilon^3). \tag{5.4b}$$

The errors in the trial functions (5.2*a*, *b*) are $O(\epsilon)$, which implies that the errors in (5.3) and (5.4) are at most $O(\epsilon^2)$ ($O(\epsilon^3)$ in (5.4) refers to the terms neglected in the expansion of (5.3)) by virtue of the variational principle. This prediction is confirmed by a comparison of the first two terms in the expansion (5.4) with (4.6*a*) and (4.7*a*).

Letting $\kappa \downarrow 0$ in (4.2) and (5.3) and integrating by parts, we obtain the shallow-water approximations

$$\lambda = \left(\frac{1+\nu^2}{2\nu}\right)\left[1 + \int_0^\infty e^{-2(\nu/\epsilon)\eta} H''(\eta) \, d\eta\right], \quad \nu = (1 - \epsilon\lambda)^{\frac{1}{2}}, \tag{5.5a, b}$$

wherein error factors of $1 + O(\kappa^2)$ are implicit.

## 6. Examples

The simplest smooth profile that satisfies (1.4) is

$$h = h_1(1 - e^{-\sigma y/h_1}), \tag{6.1}$$

for which (1.6), (4.4), (4.6*b*), (4.7*b*) and (5.4) yield

$$H = 1 - e^{-\eta}, \quad F_0 = 0, \quad F_1 = \tfrac{1}{6}\kappa^2(e^{-\eta} + e^{-2\eta}), \tag{6.2a–c}$$

$$\lambda = 1 - \tfrac{1}{2}\epsilon + \epsilon^2(\tfrac{1}{8} - \tfrac{1}{2}\kappa^2) + O(\epsilon^3), \quad \nu = 1 - \tfrac{1}{2}\epsilon + \epsilon^2(\tfrac{1}{8} - \tfrac{1}{6}\kappa^2) + O(\epsilon^3). \tag{6.3a, b}$$

Shallow-water theory yields (Ball 1967)

$$\lambda = \nu = [(1 + \tfrac{1}{4}\epsilon^2)^{\frac{1}{2}} - \tfrac{1}{2}\epsilon] \, [1 + O(\kappa^2)], \tag{6.4}$$

in agreement with (6.3) within the indicated error factors. The shallow-water approximations (5.5$a$, $b$) yield (6.4) without further approximation.

An example for which $h(y)$ is not smooth is provided by

$$h = \begin{matrix} \sigma y \\ h_1 \end{matrix} \quad (\sigma y \lessgtr h_1). \tag{6.5}$$

The counterparts of (6.2) and (6.3) are

$$H = \begin{matrix} \eta \\ 1 \end{matrix}, \quad F_0 = \begin{matrix} -\frac{1}{2} \\ 0 \end{matrix}, \quad F_1 = \begin{matrix} -\frac{1}{8}+\frac{1}{3}\kappa^2 \\ 0 \end{matrix} + (\tfrac{1}{4}-\tfrac{1}{6}\kappa^2)\,\delta(\eta-1) \quad (\eta \lessgtr 1), \tag{6.6$a$–$c$}$$

where $\delta$ is Dirac's delta function,

$$\lambda = 1+\epsilon^2(\tfrac{1}{8}-\tfrac{1}{2}\kappa^2)+O(\epsilon^3), \quad \nu = 1-\tfrac{1}{2}\epsilon-\epsilon^2(\tfrac{1}{8}+\tfrac{1}{6}\kappa^2)+O(\epsilon^3). \tag{6.7$a$, $b$}$$

Shallow-water theory yields (Minzoni 1976)

$$\lambda = 1-\mathrm{e}^{-2/\epsilon}+O(\mathrm{e}^{-4/\epsilon}), \quad \nu = (1-\epsilon\lambda)^{\frac{1}{2}}, \tag{6.8$a$, $b$}$$

within implicit error factors of $1+O(\kappa^2)$. It follows that (6.7$a$) is in error at $O(\epsilon^2)$, presumably in consequence of the discontinuity in slope.

## Appendix. Comparison with Shen, Meyer & Keller

Shen *et al.*'s (1968) equations (7) and (8) transform to [$\epsilon \rightarrow \sigma \equiv \kappa\epsilon$, $n_1 \rightarrow n$, $n_2\pi/bL \rightarrow k$, $H(X) \rightarrow h_1 H(\eta)$, $k(x) \rightarrow (kL/\lambda)^{\frac{1}{2}}\chi(\eta)$, $x \rightarrow (h_1/\sigma L)\,\eta$]

$$\int_0^\alpha [\chi^2(\eta)-1]^{\frac{1}{2}}\,\mathrm{d}\eta = (n+\tfrac{1}{2})\,\pi\epsilon \quad (n = 0, 1, 2, \ldots), \tag{A 1}$$

$$\chi(\eta)\tanh[\kappa\chi(\eta)H(\eta)] = \lambda\sigma, \quad \tanh[\kappa H(\alpha)] = \lambda\sigma. \tag{A 2$a$, $b$}$$

Expanding (A 1) and (A 2) in powers of $\sigma$, we obtain

$$\chi^2 = \epsilon\lambda H^{-1}+\tfrac{1}{3}(\sigma\lambda)^2+O(\sigma^3 H), \quad H(\alpha) = \epsilon\lambda[1+\tfrac{1}{3}(\sigma\lambda)^2+O(\sigma^4)], \tag{A 3$a$, $b$}$$

and

$$\int_0^\alpha \left[\frac{H(\alpha)-H(\eta)}{H(\eta)}\right]^{\frac{1}{2}} \mathrm{d}\eta = (n+\tfrac{1}{2})\,\pi\epsilon[1+\tfrac{1}{6}(\sigma\lambda)^2+O(\sigma^3)]. \tag{A 4}$$

(We note that (A 4) reduces to the conventional WKB approximation in the shallow-water limit, in which the $O(\sigma^2)$ terms are neglected in (A 3) and (A 4).) Letting $H$ be the variable of integration and inverting $H'$, we obtain

$$\lambda = 2n+1+\tfrac{1}{4}(2n+1)^2\,\epsilon H_0''+(2n+1)^3\,\epsilon^2[\tfrac{1}{16}(H_0'''-H_0''^2)-\tfrac{1}{6}\kappa^2]+O(\epsilon^3). \tag{A 5}$$

Comparing the $O(1)$ and $O(\epsilon)$ on the right-hand side of (A 5) with (3.6$a$) and (3.8), we find that the former is exact, whereas the second is inferior to the exact value by the factor $(2n+1)^2/[(2n+1)^2+1]$.

Substituting (6.2$a$) into (A 4) and neglecting $O(\sigma^2)$, we obtain

$$\lambda = 2n+1-(n^2+n+\tfrac{1}{4})\,\epsilon+O(\sigma^2), \tag{A 6}$$

which compares with Ball's (1967) result (which implicitly neglects $O(\sigma^2)$)

$$\lambda = (2n+1)(1+\tfrac{1}{4}\epsilon^2)^{\frac{1}{2}}-(n^2+n+\tfrac{1}{2})\,\epsilon \tag{A 7$a$}$$

$$= 2n+1-(n^2+n+\tfrac{1}{2})\,\epsilon+\tfrac{1}{8}(2n+1)\,\epsilon^2+O(\epsilon^3). \tag{A 7$b$}$$

It follows that (A 5) fails at $O(\epsilon^2)$, but this appears to reflect the intrinsic order of Keller's geometrical-optics approximation rather than the neglect of the singularity at $h = 0$.

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